Bloch Sphere and Pauli matrices

- **Bloch Sphere** representation of single qubit: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$.
  - Let $|\alpha| = \cos\frac{\theta}{2}, |\beta| = \sin\frac{\theta}{2} \iff \theta = 2\arctan\left(\frac{|\beta|}{|\alpha|}\right)$
  - $|\psi\rangle = \cos\frac{\theta}{2}e^{i\gamma}|0\rangle + \sin\frac{\theta}{2}e^{i\eta}|1\rangle \cong \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i(\eta-\gamma)}|1\rangle$.
  - Let $\phi = \eta - \gamma$. We have $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \cong \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$
    - $\theta = 2\arctan\left(\frac{|\beta|}{|\alpha|}\right), \phi = \arg\beta - \arg\alpha, (\theta,\phi) \mapsto (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
    - θ is angle between Bloch vector and $+z$, and ϕ is that between Bloch vector and $+x$
  - $$|\psi\rangle = \begin{bmatrix} |0\rangle \\ |1\rangle \\ (|0\rangle + |1\rangle)/\sqrt{2} \\ (|0\rangle - |1\rangle)/\sqrt{2} \\ (|0\rangle + i|1\rangle)/\sqrt{2} \\ (|0\rangle - i|1\rangle)/\sqrt{2} \end{bmatrix} \implies \theta = \begin{bmatrix} 0 \\ \pi \\ \pi/2 \\ \pi/2 \\ \pi/2 \\ \pi/2 \end{bmatrix}, \phi = \begin{bmatrix} ? \\ ? \\ 0 \\ \pi \\ \pi/2 \\ 3\pi/2 \end{bmatrix} \mapsto \begin{bmatrix} +z \\ -z \\ +x \\ -x \\ +y \\ -y \end{bmatrix}$$

- **Pauli matrices** constitute a basis for Hermitian and unitary operators over $\mathbb{C}^2$
  - $I = \sigma_0 \qquad X = \sigma_1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \qquad Y = \sigma_2 = \begin{bmatrix} & -i \\ i & \end{bmatrix} \qquad Z = \sigma_3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$
  - $\sigma_{A\in\{X,Y,Z\}}$ have eigenvectors corresponding to $\pm A$ (see above) with eigenvalues ± 1
  - $[\sigma_i, \sigma_j] = i\sum \epsilon_{jkl}\sigma_l$ where $\epsilon_{jkl} = 1 \iff jkl \in \{123, 231, 312\}$
- $\{I, X, Y, Z\}$ constitute orthonormal basis over SU_2 with inner product of flattened matrices
  - $\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ is unitary

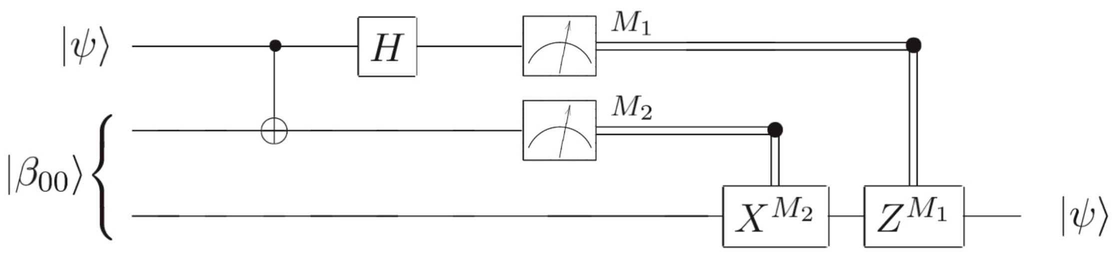
- Retrieving quantum state $|\psi\rangle \in \mathbb{C}^2$ from Bloch sphere representation $v \in \mathbb{R}^3$:
  - **Dot product between Bloch vector and Pauli matrices** $v \cdot \sigma \equiv \sum v_i\sigma_i$
  - $v \cdot \sigma = \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}$ Hermitian and unitary. $\det(v\cdot\sigma) = -1, \mathrm{tr}(v\cdot\sigma) = 0$

- Then $v \cdot \sigma = |\psi\rangle\langle\psi| - |\bar{\psi}\rangle\langle\bar{\psi}|$ and $|\psi\rangle\langle\psi| + |\bar{\psi}\rangle\langle\bar{\psi}| = I$, $\langle\bar{\psi}|\psi\rangle = 0$
- Then $|\psi\rangle\langle\psi| = \dfrac{I + v \cdot \sigma}{2}$ where $|\psi\rangle$ is the eigenvector of $v \cdot \sigma$ with eigenvalue 1
- $f(\theta v \cdot \sigma) = f(\theta)\dfrac{I + v \cdot \sigma}{2} + f(-\theta)\dfrac{I - v \cdot \sigma}{2} = \dfrac{f(\theta) + f(-\theta)}{2} I + \dfrac{f(\theta) - f(-\theta)}{2} v \cdot \sigma$
- Note the factor of 2: θ difference on Bloch sphere $\iff$ $\theta/2$ in Hilbert space

Quantum Teleportation and Superdense Coding

- **Bell Basis** $B = \left[|\beta_{00}\rangle \quad |\beta_{01}\rangle \quad |\beta_{10}\rangle \quad |\beta_{11}\rangle\right] = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}$ (exercise 4.33)
  - $B^{-1} = B^{\dagger} = (H \otimes I)C^X$ — convert between measurement in bell and computational basis
- Up to global phase factor, single-qubit application of $I, X, Z, iY = XZ$ permutes the bell basis
  - For $L = \begin{array}{c} \\ \begin{bmatrix} \begin{array}{c|cccc} & \mathbf{00} & \mathbf{01} & \mathbf{10} & \mathbf{11} \\ \mathbf{I} & 00 & 01 & 10 & 11 \\ \mathbf{X} & 01 & 00 & -11 & -10 \\ \mathbf{Z} & 10 & 11 & 00 & 01 \\ \mathbf{iY} & 11 & 10 & -01 & -00 \end{array} \end{bmatrix} \end{array}, R = \begin{bmatrix} \begin{array}{c|cccc} & \mathbf{00} & \mathbf{01} & \mathbf{10} & \mathbf{11} \\ \mathbf{I} & 00 & 01 & 10 & 11 \\ \mathbf{X} & 01 & 00 & 11 & 10 \\ \mathbf{Z} & 10 & -11 & 00 & -01 \\ \mathbf{iY} & -11 & 10 & -01 & 00 \end{array} \end{bmatrix}$
  - $\left([I, X, iY, Z]_j \otimes I\right)|\beta_k\rangle = |\beta_{L_{jk}}\rangle$, $\left(I \otimes [I, X, iY, Z]_j\right)|\beta_k\rangle = |\beta_{R_{jk}}\rangle$
  - X flips the second bit, Z flips the first bit, and $iY = ZX$ flips both bits
  - There is something special about this permutation! Specifically about how it reveals no information without having the other qubit
  - Moreover, $(s, O) \mapsto s' \iff (s', O) \mapsto s$: this is because the gates correspond to flipping
  - Generalized Clifford algebra
- Up to global phase factor, it does not matter to which qubit we apply I, X, iY, Z
  - $\forall j, k : \left([I, X, iY, Z]_j \otimes I\right)|\psi_k\rangle \cong \left(I \otimes [I, X, iY, Z]_j\right)|\psi_k\rangle$
  - Corollary: measurement of spin along any axis v of $|\beta_{11}\rangle$ yields 0
    - $\forall v \in \mathbb{R}^3, \langle\beta_{11}|\dfrac{v \cdot \sigma \otimes I + I \otimes v \cdot \sigma}{2}|\beta_{11}\rangle = 0$
    - See L, R above: $\forall j, \left([X, iY, Z]_j \otimes I\right)|\beta_{11}\rangle + \left(I \otimes [X, iY, Z]_j\right)|\beta_{11}\rangle = 0$

- Are there generalizations of this phenomenon for multiple-qubits / qutrits, etc? Are there other unique basis with corresponding operators in two-qubits (unique as in not the same up to unitary transformation)
  - No. You need maximally entangled states $\mathrm{tr}_A(\rho^{AB}) = I/2$ and they're bell basis up to equivalence
- **Superdense coding**: assume shared EPR pair in $|\psi_{00}\rangle$, convey 2 bits with 1 qubit
  - Let O denote the list $[I, X, iY, Z]$. If we need to transmit $s \in \{0,1\}^2$ then Alice applies $O_s \otimes I$, and Bob recovers s by measuring $(O_s \otimes I)|\psi\rangle$ in bell basis
  - Remark: Superdense coding uses permutation property **(exercise 2.70)**
- **Qubit teleportation**: transport unknown $|\psi\rangle$ using classical channel and shared EPR pair



  - Suppose $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, then $|\psi\rangle|\beta_{00}\rangle \mapsto_{CX} \alpha|0\rangle|\beta_{00}\rangle + \beta|1\rangle|\beta_{01}\rangle$
  - $\mapsto_H \alpha(|0\rangle + |1\rangle)|\beta_{00}\rangle + \beta(|0\rangle - |1\rangle)|\beta_{01}\rangle \cong \sum |M_z M_x\rangle(Z^{M_z}X^{M_x}|\psi\rangle)$
- **Bell's inequality**
  - Suppose $P, Q, R, S = \pm 1$ are measurements outcomes of distinct physical properties, then $QS + RS + RT - QT = (Q+R)S + (R-Q)T = \pm 2$ and $\mathbb{E}[QS + RS + RT - QT] = \mathbb{E}[QS] + \mathbb{E}[RS] + \mathbb{E}[RT] - \mathbb{E}[QT] \leq 2$
    - Remark: by $\mathbb{E}[a+b] = \mathbb{E}[a] + \mathbb{E}[b]$ we assumed *independence*
- Now, consider measurements $Q = Z \otimes I,\ \ R = X \otimes I,\ \ S = I \otimes \dfrac{-Z-X}{\sqrt{2}},\ \ T = I \otimes \dfrac{Z-X}{\sqrt{2}}$,

  on $|\beta_{11}\rangle$ we have $\langle QS\rangle = \langle RS\rangle = \langle RT\rangle = -\langle QT\rangle = 2\sqrt{2}$
- **Realism**: physical properties have values Q, R, S, T independent of observation. **Locality**: events sufficiently far apart does not influence each other—one of them has to be wrong!