Quick Linear Algebra Quick Review:

- The **adjoint** $T^* : W \to V$ of $T : V \to W$ satisfies $\langle Tv, w\rangle = \langle v, T^*w\rangle$
- An **operator** $T : V \to V$ is **self-adjoint / Hermitian** if $T = T^*$ and **normal** if $T^*T = TT^*$
- Implications of normality:
  - $\forall v : \|Tv\| = \|T^*v\|$: $T^*T = TT^* \iff \langle T^*Tv, v\rangle = \langle TT^*v, v\rangle$
  - $(v, \lambda) \in \text{Eigen}(T) \iff (v, \bar{\lambda}) \in \text{Eigen}(T^*)$: $0 = \|(T - \lambda I)v\| = \|(T - \lambda I)^*v\|$
  - $\{(v, \alpha), (w, \beta)\} \subset \text{Eigen}(T) \implies \big((\alpha \neq \beta) \iff \langle v, w\rangle = 0\big)$
    - $\alpha\langle v, w\rangle = \langle Tv, w\rangle = \langle v, T^*w\rangle = \langle v, \bar{\beta}w\rangle = \beta\langle v, w\rangle$
  - Spectral theorem: Normality $T^*T = TT^* \iff$ orthonormal eigenbasis
- An operator T is **positive** if *self-adjoint* and $\langle Tv, v\rangle \geq 0$; characterizations:
  - All nonnegative eigenvalues; has positive square root such that $R^2 = R^*R = T$
- An operator S is an **isometry / unitary** if $\forall v : \|Sv\| = \|v\|$; characterizations
  - $\forall u, v \in V : \langle Su, Sv\rangle = \langle u, v\rangle \quad S^*S = SS^* = I \quad S^{-1} = S^*$
- **Simultaneous Diagonalization Theorem**: Given Hermitian operators A, B,

  $[A, B] = 0 \iff \exists\{|i\rangle\} : A = \sum \alpha_i |i\rangle; B = \sum \beta_i |i\rangle$
  - Consider for all eigenspaces (V_i, λ_i) of A and any $|i\rangle \in V_i$, now

    $A\left(B|i\rangle\right) = BA|i\rangle = \lambda_i B|i\rangle \implies B|i\rangle \in V_i \implies B_{|V_i} \in \mathscr{L}(V_i)$. Now $B_{|V_i}$ is Hermitian

    and has spectral decomposition on V_i, then $A_{|V_i}, B_{|V_i}$ are simultaneously diagonalizable.
- **Polar Decomposition**: $\forall A \in \mathscr{L}(V), \exists U : U^\dagger U = I, A = U\sqrt{A^\dagger A} = \sqrt{AA^\dagger}U$
  - Assume for convenience A invertible: let $\sqrt{A^\dagger A} = \sum \lambda_i |i\rangle\langle i|$ and $|\psi_i\rangle = \dfrac{A|i\rangle}{\lambda_i}$. Now

    $\{|\psi_i\rangle\}$ is an orthonormal basis since $\langle \psi_i|\psi_j\rangle = \dfrac{\langle j|A^\dagger A|i\rangle}{\lambda_i\lambda_j} = \delta_{ij}$. Let $U = \sum |\psi_i\rangle\langle i|$,

    then $U\sqrt{A^\dagger A}|i\rangle = \lambda_i^2 U|i\rangle = \lambda_i^2 \dfrac{A|i\rangle}{\lambda_i^2} \implies U\sqrt{A^\dagger A} = A$.
  - Now assume $A = KU, A = U\sqrt{A^\dagger A} = U\sqrt{A^\dagger A}U^\dagger U \implies K = U\sqrt{A^\dagger A}U^\dagger$ is positive.
  - At the same time $A = KU \implies AA^\dagger = K^2 \implies K = \sqrt{AA^\dagger}$
  - **Singular Value Decomposition**: $\forall A \in \mathscr{L}(V), \exists U, V : U^\dagger U = V^\dagger V = I, A = UDV$
    - Let $\sqrt{A^\dagger A} = TDT^\dagger$, then $A = S\sqrt{A^\dagger A} = (ST)DT^\dagger$; T and S unitary and D diagonal

- Given $T = \sum \lambda_i | i \rangle \langle i | \in \mathscr{L}(V)$ and $f : \mathbb{F} \to \mathbb{F}$, **Sylvester formula** extends f to a function on *diagonalizable operators* over V defined by $f(T) = \sum f(\lambda_i) | i \rangle \langle i |$
- **Outer products**: the outer-product $| v \rangle \langle w |$ is the operator such that $(| w \rangle \langle v |) | v' \rangle = \langle v | v \rangle | w \rangle$
  - Completeness relation: let $| i \rangle$ be an orthonormal basis and consider the operator $\sum | i \rangle \langle i |$. Note that $\left(\sum | i \rangle \langle i | \right) | v \rangle = | v \rangle \implies \sum | i \rangle \langle i | = I$
  - Consider operator $A : V \to W$ with orthonormal bases $| v_j \rangle, | w_i \rangle$ respectively, then $A = I_W A I_V = \sum | w_i \rangle \langle w_i | A | v_j \rangle \langle v_j | = \sum \left(\langle w_i | A | v_j \rangle \right) | w_i \rangle \langle v_j |$.
  - Now $\mathscr{M}(A)_{ij} = \langle w_i | A | v_j \rangle$: $A | v_j \rangle$ singles j-th column, while $\langle w_i | \left(A | v_j \rangle \right)$ singles i-th entry
  - By corollary, outer product representation $A = \sum \mathscr{M}(A)_{ij} | w_i \rangle \langle v_j |$
- Given bases $| i \rangle$ and $| j \rangle$ for spaces V, W, the **tensor product** space $V \otimes W$ is spanned by $| i \rangle \otimes | j \rangle \equiv | i \rangle | j \rangle$
  - Tensor product of operators: $(A \otimes B) | i \rangle | j \rangle \equiv \left(A | i \rangle \right) \otimes \left(B | j \rangle \right)$
  - **Kronecker Product** $A \otimes B = \begin{bmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \dots & A_{mn}B \end{bmatrix}$ e.g. $\sigma_X \otimes \sigma_Y = \begin{bmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{bmatrix}$
  - Extending inner product: $\left(| w_1 \rangle | v_1 \rangle \right)^\dagger \left(| w_2 \rangle | v_2 \rangle \right) = \langle v_1 | v_2 \rangle \langle w_1 | w_2 \rangle$
- Given a Hilbert space V over field $\mathbb{C}$, The **Hilbert-Schmidt** inner product (or **trace inner product**) on operators $\langle \cdot , \cdot \rangle : \mathscr{L}(V) \times \mathscr{L}(V) \to \mathbb{C}$ is defined via $\langle A, B \rangle \equiv \mathrm{tr}(A^\dagger B)$

Postulates of Quantum Mechanics

- Postulate 1: Systems described by unit vectors in **Hilbert** space (complex, with inner products)
- Postulate 2: Evolution of *closed* system described by **Schrödinger Equation**
  $i\hbar\frac{d|\Psi\rangle}{dt} = H|\Psi\rangle$, H being a h*ermitian* operator denoting the **Hamiltonian** of the system
  - Mechanically for 1D particle, H is associated the operator $H \equiv -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V$
  - *Assuming time-constant* H, $H = \sum E|E\rangle\langle E|$ where $|E\rangle$ denote stationary states with unchanging, definite energy values, and E is the corresponding total energy of the system.
  - Solution: $|\Psi\rangle = e^{-iHt/\hbar}|\Psi_0\rangle = \left(\sum e^{-iEt/\hbar}|E\rangle\langle E|\right)|\Psi_0\rangle = \sum \langle E|\Psi_0\rangle e^{-iEt/\hbar}|E\rangle$
- Postulate 3: Measurements with outcomes $\{m\}$ described via collection of operators $\{M_m\}$ *satisfying completeness relation* $\sum M_m^\dagger M_m = I$. For system in state $|\Psi\rangle$,
  $p(m) = \langle\Psi|M_m^\dagger M_m|\Psi\rangle$, and $|\Psi\rangle \mapsto \frac{M_m|\psi\rangle}{\sqrt{p(m)}}$
- Postulate 4: composite system with subsystems in states $|\Psi_1\rangle, |\Psi_2\rangle$ has state $|\Psi_1\rangle \otimes |\Psi_2\rangle$
  - Equivalent with **superposition principle**: imagine two qubits, if each cubit is allowed $\alpha|0\rangle + \beta|1\rangle$, then the composite system should be allowed $\sum \alpha_{ij}|i\rangle|j\rangle$

Examples and Equivalent Characterizations

- **Measurement in computational basis:** Suppose $|\Psi\rangle = \sum c_i|i\rangle$, then $\{|i\rangle\langle i|\}$ are measurement operators satisfying completeness relation and $P(i) = c_i^2$. After outcome i we have $|\Psi\rangle \mapsto \frac{\langle\Psi|i\rangle\langle i|i\rangle\langle i|\Psi\rangle}{c_i} = |i\rangle$
- **Projective Measurements**: Described a *Hermitian* operator $M = \sum m|m\rangle\langle m|$. Then $p(m) = \|\langle\psi|m\rangle\|^2$ and $|\Psi\rangle \mapsto |m\rangle$
  - Then $\mathbb{E}[M] = \langle\psi|M|\psi\rangle$, and $\sigma_M^2 = \langle(M - \langle M\rangle)^2\rangle = \langle M^2\rangle - \langle M\rangle^2$
  - Corollary: commuting projective observables are simultaneously measurable
- **Heisenberg Uncertainty Principle**: Given Hermitian observable operators A, B and $|\Psi\rangle$.
  - Let $\langle\Psi|AB|\Psi\rangle = x + iy$, then $\langle\Psi|[A,B]|\Psi\rangle = 2iy$, $\langle\Psi|\{A,B\}|\Psi\rangle = 2x$
  - $|\langle\Psi|[A,B]|\Psi\rangle|^2 + |\langle\Psi|\{A,B\}|\Psi\rangle|^2 = 4|\langle\Psi|AB|\Psi\rangle|^2$

- Now, plug $A|\Psi\rangle$ and $B|\Psi\rangle$ into Cauchy-Schwarz inequality and $A = A^\dagger, B = B^\dagger$ yields $|\langle\Psi|AB|\Psi\rangle|^2 \leq \langle\Psi|A^2|\Psi\rangle\langle\Psi|B^2|\Psi\rangle \implies \langle\Psi|[A,B]|\Psi\rangle|^2 \leq 4\langle\Psi|A^2|\Psi\rangle\langle\Psi|B^2|\Psi\rangle$
- Substitute $A \mapsto C - \langle C\rangle, B \mapsto D - \langle D\rangle$ yields $\Delta(C)\Delta(D) \geq \dfrac{|\langle\Psi|[C,D]|\Psi\rangle|}{2}$

- **Positive Operator-Valued Measure (POVM) Measurements**:
    - Consider general measurement operators $\{M_m\}$ and let $E_m \equiv M_m^\dagger M_m$
    - Then completeness relation requires $\sum E_m = I$ and $p(m) = \langle\Psi|E_m|\Psi\rangle$
    - When post-measurement state is of little interest
- $e^{i\theta}|\Psi\rangle$ is equivalent with $|\Psi\rangle$ up to **global phase factor** $e^{i\theta}$
    - Global phase factors do not affect measurement outcomes
- Two *amplitudes* $\alpha, \beta \in \mathbb{C}$differ by a **relative phase** *in some basis* if $\alpha = \exp(i\theta)b$. Two *states* differ by relative phase if *each* of the amplitudes in the basis differ by phase factor.
- The composite system of $A \times B$ is in an **entangled** state if $|\Psi_{A\times B}\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle$
    - Cannot be decomposed into tensor product of subsystems, analogous to having full rank
    - Specifying n-bit classical system takes n bits, while requires $O(2^n)$ for entangled n-qubits
- **Projective Measurement + Unitary Evolution $\iff$ General Measurements**
    - Equivalent procedure to performing $\{M_m\}$ on Q (with same probability and resulting state)
    - Introduce ancilla system with state space M and orthonormal basis $|m\rangle \leftrightarrow M_m$
    - Define unitary $U : U|\psi\rangle|0\rangle = \sum M_m|\psi\rangle|m\rangle$. Now using completeness relation we have $\langle\phi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \sum \langle\phi|M_m^\dagger M_{m'}|\psi\rangle\langle m|m'\rangle = \langle\phi|\sum M_m^\dagger M_{m'}|\psi\rangle = \langle\phi|\psi\rangle$
        - Extend U from $Q \times \{|0\rangle\}$ to $Q \times M$
    - Now perform projective measurement $P_m = I_Q \otimes |m\rangle\langle m|$ on QM with $U|\psi\rangle|0\rangle$, then $P_m U|\psi\rangle|0\rangle = I_Q \otimes \left(|m\rangle\langle m|\right)\sum M_{m'}|\psi\rangle|m'\rangle = \sum \left(M_{m'}|\psi\rangle\right) \otimes \left(\langle m|m'\rangle|m\rangle\right) = \left(M_m|\psi\rangle\right) \otimes |m\rangle$ and $p(m) = \langle\psi|M_m^\dagger M_m|\rangle\langle m|m\rangle$ and $U|\psi\rangle|0\rangle \mapsto \dfrac{\left(M_m|\psi\rangle\right) \otimes |m\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$

Density Operators and Ensembles

- A quantum system in an **ensemble of pure states** $\{p_i, |\psi_i\rangle\}$ (in $|\psi_i\rangle$ with probability p_i) is described by the **density operator** $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$
  - Note that while $\rho = |\psi\rangle\langle\psi|$ for $|\psi\rangle = \sum \sqrt{p_i}|\psi_i\rangle$, it *is not* a unit vector
  - Constraints: $\mathrm{tr}(\rho) = 1 \iff$ law of total probability and $\lambda_\rho \geq 0 \iff$ nonnegative probability. Then the Eigen-stuff of ρ gives one of ρ's *possible* ensembles (see below)
- Postulate 2: Density operator evolves unitarily by $\rho \xrightarrow{U} \sum p_i U^\dagger |\Psi_i\rangle\langle\Psi_i| U = U^\dagger \rho U$
- Postulate 3: Measurement denoted by collection $\{M_m\}$ obeying $\sum M_m^\dagger M_m = I$
  - $p(m \,|\, |\psi\rangle = |\psi_i\rangle) = \mathrm{tr}(M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|) \implies p(m) = \sum p(m\,|\,i)p_i = \mathrm{tr}(M_m \rho M_m^\dagger)$
  - Similarly, $\rho \mapsto \rho_m = \dfrac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)}$
- ρ denotes a **pure state** if $\rho = |\Psi\rangle\langle\Psi|$ for some unit $|\Psi\rangle$, else it is in a **mixed state**
  - ρ pure $\iff$ $\mathrm{tr}(\rho^2) = 1$**:** Let $\lambda_i \in \mathbb{R}$ be eigenvalues of ρ, $\sum \lambda_i = 1$, then $\sum \lambda_i^2 = 1 \implies \exists j : \lambda_i = \delta_{ij}$
- Example of using mixed states: *If result of a measurement is lost* then our best guess is

$$\rho \mapsto \sum p(m)\rho_m = \sum \mathrm{tr}(M_m^\dagger M_m \rho)\frac{M_m \rho M_m^\dagger}{\mathrm{tr}(M_m^\dagger M_m \rho)} = \sum M_m \rho M_m^\dagger$$

- Unitary freedom in ensemble of density matrices
  - Different ensembles of quantum states may give rise to the same density matrix!
  - For convenience, say that (not-necessarily unit) $|\tilde{\psi}_i\rangle$ *generates* ρ if $\rho = \sum |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i|$
    - In the ensemble picture where $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$, $|\tilde{\psi}_i\rangle = \sqrt{p_i}|\psi_i\rangle$
  - **Theorem: $|\tilde{\psi}_i\rangle, |\tilde{\phi}_j\rangle$ generate the same density matrix $\iff$ $\Psi = \Phi U$ for unitary U**
    - Now $\Psi = \left[|\tilde{\psi}_1\rangle, \ldots, |\tilde{\psi}_n\rangle\right]$ generates density matrix $\Psi\Psi^\dagger = (\Phi U)(\Phi U)^\dagger = \Phi\Phi^\dagger$
- **Bloch sphere representation of mixed states**: $\rho = \dfrac{I + v \cdot \sigma}{2}$
  - Consider the mapping ϕ from rank-1 density matrix to generalized Bloch vector

$$|\psi\rangle\langle\psi| = \frac{\phi(|\psi\rangle\langle\psi|) \cdot \sigma}{2} \text{ with } \sigma \equiv [I \quad X \quad Y \quad Z]$$

    - Remark: ϕ is linear since the relation $n \mapsto n \cdot \sigma$ is linear

- Then $\rho = \sum p_i |\psi_i\rangle\langle\psi_i| = \dfrac{I + v \cdot \sigma}{2}$ for $v = p_i \vec{n}_i$

- ρ pure $\iff \|v\| = 1$: 2ρ has characteristic eq $(1 + v_3 - \lambda)(1 - v_3 - \lambda) - v_1^2 - v_2^2 = 0$

  $\iff \lambda^2 - 2\lambda + 1 - \|v\| = 0$ implies *eigenvalues for ρ are* $\dfrac{1}{2}\left(1 \pm \|v\|\right)$, then

  $4\mathrm{tr}(\rho^2) = (1 + \|v\|)^2 + (1 - \|v\|)^2 = 2 + \|v\|^2 \implies \mathrm{tr}(\rho^2) = \dfrac{1}{2}(1 + \|v\|)$

Reduced density operator

- Let $\mathscr{H}(V) \subsetneq \mathscr{L}(V)$ be the subset of Hermitian operators. The **partial trace** over B is the *linear map* $\text{tr}_B : \mathscr{H}(A \otimes B) \to \mathscr{H}(A)$ satisfying $\text{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = \text{tr}(|b_1\rangle\langle b_2|)|a_1\rangle\langle a_2|$
- Given $\rho^{AB} \in \mathscr{H}(A \otimes B)$, the **reduced density operator** for system A is $\text{tr}_B(\rho^{AB})$.
  - In particular, $\rho^{AB} = \sum \alpha_{ij}\left(|a_i\rangle\langle a_i| \otimes |b_j\rangle\langle b_j|\right) \implies \text{tr}_B(\rho^{AB}) = \sum \alpha_{ij}|a_i\rangle\langle a_i| = \sum_j \langle b_j|\rho|b_j\rangle$
- Theorem: partial trace is the *unique* linear map satisfying $\text{tr}((M \otimes I)\rho^{AB}) = \text{tr}(M\text{tr}_B(\rho^{AB}))$
  - Corollary: measuring ρ^{AB} against $M \otimes I$ equivalent to measuring $\rho^A = \text{tr}_B(\rho^{AB})$ against M
  - Proof of property: given eigenvalue decomposition $\rho^{AB} = \sum \alpha_{ij}|a_i\rangle\langle a_i| \otimes |b_j\rangle\langle b_j|$,

    $$\text{tr}\left((M \otimes I)\rho^{AB}\right) = \text{tr}\left(\sum \alpha_{ij}\left(M|a_i\rangle\langle a_i|\right) \otimes |b_j\rangle\langle b_j|\right) = \text{tr}\left(M\sum \alpha_{ij}|a_i\rangle\langle a_i|\right) = \text{tr}\left(M\text{tr}_B(\rho^{AB})\right)$$
  - Proof of uniqueness: Assume f satisfies $\text{tr}\left((M \otimes I)\rho\right) = \text{tr}(Mf(\rho))$. Consider the orthonormal basis $\{M_i = |a_i\rangle\langle a_i|\}$ over $\mathscr{H}(A)$ with trace inner product, Fourier expansion:

    $$f(\rho) = \sum \langle f(\rho), M_i\rangle M_i = \sum \text{tr}\left(f(\rho)^\dagger M_i\right)M_i = \sum \text{tr}\left(M_i f(\rho)\right)M_i = \sum \text{tr}\left((M_i \otimes I)\rho\right)M_i$$

    Given spectral decomposition $\rho = \sum \alpha_{ij}\left(|a_i\rangle\langle a_i| \otimes |b_j\rangle\langle b_j|\right)$, we have

    $$f(\rho) = \sum |a_i\rangle\langle a_i|\text{tr}\left(\left(|a_i\rangle\langle a_i| \otimes I\right)\rho\right) = \sum \alpha_{ij}|a_i\rangle\langle a_i| = \text{tr}_B(\rho)$$
- **Schmidt Decomposition**: Given *pure* state $|\psi\rangle$ for composite system AB, there exists orthonormal basis $\{|i_A\rangle\}, \{|i_B\rangle\}$ for systems A, B such that $|\psi\rangle = \sum \lambda_i|i_a\rangle|i_b\rangle$ and $\sum \lambda_i^2 = 1$. $\{\lambda_i\}$ are **Schmidt coefficients**, $\{|i_A\rangle, |i_B\rangle\}$ are the **Schmidt bases**, and **Schmidt number** is number of nonzero Schmidt coefficients
  - Corollary: given pure state $|\psi\rangle$, $\text{tr}_A(|\psi\rangle\langle\psi|)$ and $\text{tr}_B(|\psi\rangle\langle\psi|)$ have same eigenvalues λ_i^2
  - Proof: $|\psi\rangle = \sum_{jk} A_{jk}|j\rangle|k\rangle$. By SVD $A = UDV \implies A_{jk} = \sum_i U_{ji}D_{ii}V_{ik}$ and

    $$|\psi\rangle = \sum_{ijk} U_{ji}D_{ii}V_{ik}|j\rangle|k\rangle = \sum_i D_{ii}\left(\sum_j U_{ji}|j\rangle\right)\left(\sum_k V_{ik}|k\rangle\right) = \sum_i \lambda_i|i_A\rangle|i_B\rangle$$
  - Corollary: Schmidt number invariant under unitary transformations $U = U_A \otimes U_B$
    - Replace $|i_A\rangle \mapsto U_A|i_A\rangle, |i_B\rangle \mapsto U_B|i_B\rangle$
  - $|\psi\rangle, |\phi\rangle$ over AB have same Schmidt coefficients $\iff \exists U_A, V_B : |\psi\rangle = (U_A \otimes V_B)|\phi\rangle$
  - $|\psi\rangle$ is product state $\iff$ it has Schmidt number 1
    - $|\psi\rangle = \sum \lambda_i|i_A\rangle|i_B\rangle$ then $\text{tr}_A(|\psi\rangle\langle\psi|) = \sum \lambda_i^2|i_A\rangle\langle i_A|$ pure $\iff \sum \lambda_i^4 = \sum \lambda_i^2 = 1$

- **Purification**: For every density operator ρ^A of system A there exists system R and product state $|AR\rangle$ such that $\rho^A = \mathrm{tr}_R(|AR\rangle\langle AR|)$. R is the *reference system*
  - For $\rho^A = \sum p_i |i_A\rangle\langle i_A|$, let R have the same state space as A with basis $|i_R\rangle$ and define $|AR\rangle = \sum \sqrt{p_i}|i_A\rangle|i_R\rangle$. Now $\mathrm{tr}_R(|AR\rangle\langle AR|) = \rho^A$ and $|AR\rangle$ is pure state since $|i_A\rangle|i_R\rangle$ are orthonormal and $\sum p_i = 1$
  - Remark: Schmidt basis of A for $|AR\rangle$ diagonalizes ρ^A
  - Unitary freedom: $\mathrm{tr}_R(|AR_1\rangle\langle AR_1|) = \mathrm{tr}_R(|AR_2\rangle\langle AR_2|) = \rho^A \implies \exists U_R : |AR_1\rangle = (I_A \otimes U_R)|AR_2\rangle$