Physics 151 Final Project

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1 Introduction

1.1 Motivation

In mechanics, a traditional approach often involves a top-down methodology: for a system comprising various subsystems, the time-evolution of the entire system is analyzed to deduce the time-evolution of its individual components. This project aims to investigate the converse inference question: can the known time-evolution of a subsystem provide insights into the time-evolution of the entire system?

In Hamiltonian mechanics, the time-evolution of an isolated system is represented by a one-parameter group of canonical transformations. Each point t in time offers a snapshot of this evolution corresponding to a specific canonical transformation. When considering a subsystem, we use the term "dynamical closure" to refer to the isolated time-evolution of a larger system that aligns with the known dynamics of the subsystem. A "snapshot closure" at a fixed time t is a canonical transformation that, when projected (in a suitable sense to be defined) onto the subsystem's phase space, matches its state at that time. To construct a dynamical closure, one must first dilate each subsystem snapshot to a canonical transformation on a larger system, then assemble these dilated transforms into a continuous one-parameter group.

Scope and Approach

Currently, our focus is on the preliminary step of characterizing mappings that can be extended into a canonical transformation. We draw inspiration from quantum theory for discrete systems with finite degrees of freedom. In this realm, a density operator map that projects to a canonical transformation is characterized as a completely-positive trace-preserving (cptp) map. Every cptp map can be extended to a canonical transformation within a system having no more than n^3 degrees of freedom. Furthermore, there is a specific criterion for the minimal dilation that can be unitarily embedded within any other compatible dilation. The feasibility of snapshot closures in quantum systems strongly suggests the possibility of similar closures in classical systems. We intend to approach this by reinterpreting Hamiltonian mechanics within the framework of Hilbert spaces, utilizing quantum theory techniques to hopefully achieve snapshot dilation.

Outline

The project is organized as follows: We start with a detailed review of Hamiltonian mechanics and Liouville's theorem, focusing particularly on the precise characterization of canonical transformations and Hamiltonian time-evolution. Liouville's theorem naturally leads us into the Koopman von Neumann theory, a reformulation of Hamiltonian mechanics with Hilbert spaces. We explore the Hilbert space formulation of classical mechanics, paying special attention to the construction of composite systems and subsystems. This approach, emphasizing parallel mathematical structures, underscores the distinctions between quantum and classical theories. We conclude with the example of a coupled harmonic oscillator system.

Summary of results

Our hope is to use a Hilbert space formulation of classical mechanics to utilize the dilation machinery for quantum systems. In the process, we find that the observable effects of a classical partial system's evolution contain less information than starting with a composite system and "forgetting" about the ancilla. This is in contrast to quantum theory, in which the appearance of noncommutative observables gives more information about how a subsystem possibly correlates with the environment. This finding highlights the key role of noncommutative observables in quantum theory, especially in relation to composite and subsystems. This also suggests that a separate, geometric approach is needed to construct the snapshot closure of classical systems.

1.2 Preliminary definitions

We begin with our definition of closure for a quantum system with discrete, finite degrees of freedom. We also enumerate some motivating results from this regime. We use $\mathcal{H}$ to denote Hilbert spaces, $T(\mathcal{H})$ the space of unit-trace positive density operators, and $B(\mathcal{H})$ the space of bounded operators, and $\mathrm{End}(S)$ the mappings from S onto itself.

Definition 1.1 (*closure*). Consider a quantum system with discrete, finite degrees of freedom. Denote its Hilbert space $\mathcal{H}_1$ and the space of density operators $T(\mathcal{H}_1)$. Given a dynamical description $E_1^{(-)} : \mathbb{R} \to \mathrm{End}(T(\mathcal{H}_1))$, a dynamical closure of $E_1^{(-)}$ consists of an ancilla system $\mathcal{H}_2$, a distinguished state $|\psi_2^0\rangle \in \mathcal{H}_2$, and a hermitian Hamiltonian operator $H \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ generating unitary time-evolution $E_{1,2}^{(-)} : \mathbb{R} \to \mathrm{End}(T(\mathcal{H}_1 \otimes \mathcal{H}_2))$ such that

$$E_1^t(\rho_1^0) = \mathrm{tr}_2 \left[E_{1,2}^t(\rho_1^0 \otimes |\psi_2^0\rangle\langle\psi_2^0|) \right]$$

Equivalently, if the following commutative diagram commutes for all t

$$\begin{array}{ccc} T(\mathcal{H}_1 \otimes \mathcal{H}_2) & \xrightarrow{E_{1,2}^t} & T(\mathcal{H}_1 \otimes \mathcal{H}_2) \\ {\scriptstyle -\otimes|\psi_2^0\rangle\langle\psi_2^0|}\uparrow & & \downarrow{\scriptstyle \mathrm{tr}_2} \\ T(\mathcal{H}_1) & \xrightarrow{E_1^t} & T(\mathcal{H}_1) \end{array}$$

A snapshot closure of a map $E_1^t : T(\mathcal{H}_1) \to T(\mathcal{H}_1)$ consists of a unitary $E_{1,2}^t$ such that the diagram above commutes for fixed t.

For such quantum systems we have a strong characterization of dynamical descriptions which admit a closure.

Proposition 1.1. A snapshot of density operator evolution E_1^t may be dilated to some unitary evolution $E_{1,2}^t$ if and only if it is a completely positive trace-preserving map.

The proof of the following theorem is also constructive. We leave it to the appendix.

Theorem 1.2 (*snapshot dilation for quantum systems*). The projection of every unitary evolution is a cptp map. Conversely, every cptp map $\phi : T(\mathcal{H}_1) \to T(\mathcal{H}_1)$ may be dilated to a unitary map $U : T(\mathcal{H}_2 \otimes \mathcal{H}_1) \to T(\mathcal{H}_2 \otimes \mathcal{H}_1)$ such that

$$\phi(\rho) = \mathrm{tr}_2 \left[U(\rho \otimes |0\rangle\langle 0|) U^\dagger \right]$$

For $\dim \mathcal{H}_1 = n$, the ancilla $\mathcal{H}_2$ may always be chosen such that $\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) \leq n^3$.

The definitions above have direct analogues in the classical regime: Let N_1, N_2 denote the phase spaces of two subsystems. Given the time-evolution $E_{1,2}^{(-)} : \mathbb{R} \to \mathrm{End}(N_1 \times N_2)$ of a classical Hamiltonian composite system and $x_2^0 \in N_2$, let π_1 denote projection onto the first coordinate. The dynamical description $E_1^{(-)} : \mathbb{R} \to \mathrm{End}(N_1)$ of subsystem 1 is

$$E_1^t(x_1^0) = \pi_1 \left[E_{1,2}^t(x_1^0, x_2^0) \right]$$

In terms of commutative diagrams

$$\begin{tikzcd} N_1 \times N_2 \arrow[r, "E_{1,2}^t"] & N_1 \times N_2 \arrow[d, "\pi_1"] \\ N_1 \arrow[u, "{(-,x_2^0)}"] \arrow[r, dashed, "E_1^t"] & N_1 \end{tikzcd}$$

Again, we are interested in a characterization of the dynamics of partial systems.

Definition 1.2 (*closure*). Consider a classical system with finite, continuous degrees of freedom and phase space N_1. Given a dynamical description $E_1^{(-)} : \mathbb{R} \to \mathrm{End}(N_1)$, a dynamical closure of $E_1^{(-)}$ consists of an ancilla system with phase space N_2, a distinguished point $x_2^0 \in N_2$, and a Hamiltonian function $H^{(-)} : N_1 \times N_2 \to \mathbb{R}$ which generates time-evolution $E_{1,2}^{(-)} : \mathbb{R} \to \mathrm{End}(N_1 \times N_2)$ such that

$$E_1^t(x_1^0) = \pi_1 [E_{1,2}^t(x_1^0, x_2^0)]$$

Equivalently, if the following commutative diagram commutes for all t

$$\begin{tikzcd} N_1 \times N_2 \arrow[r, "E_{1,2}^t"] & N_1 \times N_2 \arrow[d, "\pi_1"] \\ N_1 \arrow[u, "{(-,x_2^0)}"] \arrow[r, "E_1^t"] & N_1 \end{tikzcd}$$

A snapshot closure of $E_1^t : T(\mathcal{H}_1) \to T(\mathcal{H}_1)$ consists of a canonical transform $E_{1,2}^t$ such that the diagram above commutes for fixed t.

2 Koopman von Neumann theory

Koopman von Neumann theory is the formulation of Hamiltonian mechanics using Hilbert spaces. In this section, we consider a system S with finite n degrees of freedom. Its configuration manifold is denoted $M \cong \mathbb{R}^n$, and phase space $N \cong T^*M \cong \mathbb{R}^{2n}$. Points of the phase space are denoted $(q_1, \cdots, q_n, p_1, \cdots, p_n) = x \in N$. Refer to the appendix for a review of Hamiltonian mechanics. We let $d^{2n}x$ denote the volume form on N. A Hamiltonian scalar field $H : N \to \mathbb{R}$ generates time-evolution $E^t : N \to N$ by solving Hamilton's equations

$$E^t(x_0) = e^{X_H t} x_0$$

Here X_H is the Hamiltonian vector field generated by H. We may denote probabilistic knowledge of S by a a nonnegative density function $N \xrightarrow{p} \mathbb{R}_{\geq 0}$ which integrates to 1

$$\int p(x)\, d^{2n}x = 1$$

Time-evolution E^t permutes the domain N of p. We can consider p, or an arbitrarily-valued function with domain N, being "carried along" by such a flow which permutes its domain. This is made precise by the following definition

Definition 2.1 (*function pullback*). A morphism $A \xrightarrow{\varphi} B$ induces a pullback morphism

$$\hom(B, C) \xrightarrow{\varphi^*} \hom(A, C)$$

by $f \mapsto f\varphi$, as shown in the following commutative diagram

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, dashed, "\varphi^* f"'] & & B \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

Under the substitution $\varphi \mapsto N \xrightarrow{E^{-t}} N$ and $C \mapsto \mathbb{R}$, we have

$$\begin{tikzcd} N \arrow[rr, "E^{-t}"] \arrow[dr, "(E^{-t})^* p^0"'] & & N \arrow[dl, "p^0"] \\ & \mathbb{R} & \end{tikzcd}$$

The time-evolved probability distribution p^t evaluated at $x \in N$ is equivalent to the original probability distribution p^0 evaluated at $E^{-t}(x) \in N$, which evolves to x in time t.

$$p^t(x) = [(E^{-t})^* p^0](x) = p^0 \left[E^{-t}(x)\right]$$

We reserve subscripts for indexing possible subsystems and use superscripts to denote time.

Definition 2.2 (*Liouville's equation*). Given a system with phase space N and Hamiltonian $H : N \to \mathbb{R}$, A probability distribution $N \xrightarrow{p} \mathbb{R}$ on phase space N evolves according to

$$p^t = (e^{-tX_H})^* p^0 = (E^{-t})^* p^0$$

Written in differential equation form, this is equivalent to

$$\partial_t p(x) = -X_H p(x) = \{H, p\}$$

Multiply both sides by i yields $i\partial_t \rho = \hat{L}\rho$. Here we introduce the Liouvillian $\hat{L}$:

$$\hat{L} = iX_H = i\left(\partial_{q_i} H\right)\partial_{p_i} - i\left(\partial_{p_i} H\right)\partial_{q_i}$$

Definition 2.3 (*Hilbert space of phase space*). The Hilbert space $\mathcal{H}$ associated with the phase space N is the space of square-integrable functions $N \to \mathbb{C}$ with the inner product

$$\langle \phi | \psi \rangle = \int_N \phi^*(x)\psi(x)\, d^{2n}x$$

Instead of time-evolving ρ, we can also consider evolving its square root $\sqrt{\rho} : N \to \mathbb{C} = \mathcal{H}$.

$$|\psi^0(x)|^2 = \rho^0(x) \implies |\psi^t(x)|^2 = \rho^t(x)$$

This is because maps on the codomain (squaring in our case) are compatible with pullback

$$|\psi^t(x)|^2 = \left|\left[\left(E^{-t}\right)^* \psi^0\right](x)\right|^2 = |\psi^0(E^{-t}(x))|^2 = \rho^0(E^{-t}(x)) = \rho^t(x)$$

The phase can be taken to be zero. The time-evolution of root-densities gives us a map

$$U^t = \left(E^{-t}\right)^* : \mathcal{H} \to \mathcal{H} \tag{2.1}$$

Explicitly in terms of its application, for $\psi^0 \in \mathcal{H}$ we have

$$(U^t\, \psi^0)(x) = \psi^0(E^{-t}(x))$$

Proposition 2.1. Hamiltonian time-evolution $\mathcal{H} \xrightarrow{U^t} \mathcal{H}$ of root densities is unitary.

Proof: We show that U^t is surjective and preserves the inner product. Given $\psi^0, \phi^0 \in \mathcal{H}$, let ψ^t, ϕ^t denote $U^t\psi^0, U^t\phi^0$ respectively. By Liouville's theorem $(E^{-t})^* d^{2n}x = d^{2n}x$, then

$$\begin{aligned}
\langle \phi^t | \psi^t \rangle &= \int_N \phi^t(x)^* \psi^t(x)\, d^{2n}x \\
&= \int_N \phi^0(E^{-t}(x))^* \psi^0(E^{-t}(x))\, d^{2n}x \\
&= \int_N \phi^0(x)^* \psi^0(x)\ \left[(E^{-t})^* d^{2n}x\right] \\
&= \int_N \phi^0(x)^* \psi^0(x)\, d^{2n}x = \langle \phi^0 | \psi^0 \rangle
\end{aligned}$$

It is a surjection since $\phi^t = U^t(U^{-t}(\phi^t))$.

Definition 2.4 (*dual*). The dual of $|g\rangle \in \mathcal{H}$ is the linear map $g^* : \mathcal{H} \to \mathbb{C}$ defined by

$$g^* f = \langle g|f\rangle = \int g^*(x) f(x)\, dx$$

In bra-ket notation we denote it $\langle g|$. The space of continuous duals consists of such maps resulting from $f \in \mathcal{H}$ and is denoted $\mathcal{H}^*$.

The space of linear operators $L(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}^*$ over $\mathcal{H}$ maps a scalar field on phase space to another scalar field. Let $\hat{\delta}$ denote the dirac delta. Informally, we may consider the dirac deltas for each $x \in N$ forming a "basis" of $\mathcal{H}$ with scalar functions $f \in \mathcal{H}$ represented by

$$f(x) = \langle x|f\rangle = \int_{x'} \hat{\delta}(x' - x) f(x')\, dx'$$

The basis set is exactly $N \cong \mathbb{R}^{2n}$: $\mathcal{H}$ is an infinite-dimensional Hilbert space. Every linear operator $A \in L(\mathcal{H})$ may be expanded in this basis as

$$A = \int\int A(x, x')|x\rangle\langle x'|\, dx\, dx'$$

Fixing the dirac delta basis, $A \in L(\mathcal{H})$ may be represented in terms of its matrix elements $A(x, x') = \langle x|A|x'\rangle$. We interchangeably denote A by its kernel $A : N \times N \to \mathbb{C}$. The application of A on a scalar function f is defined by

$$A|f\rangle = \int_x \int_{x'} dx\, dx'\ [A(x, x')|x\rangle\langle x'|f\rangle] = \int_x \int_{x'} dx\, dx'\ [A(x, x') f(x')|x\rangle]$$

The function representation of $A\, f$ is an integral transform

$$(A\, f)(x) = \int dx'\, A(x, x') f(x')$$

The complex conjugate of A is defined by conjugate-transposing the matrix elements.

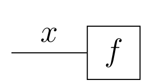
Proposition 2.2. The kernel of the unitary generated by pullback along canonical transform E^{-t} as in equation 2.1 is

$$U^t(x, x') = \hat{\delta}(x - E^t(x')) \tag{2.2}$$

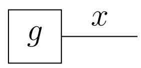
This can be verified by explicitly evaluating the integral transform

$$\int dx'\, U(x, x')\psi(x) = \int dx'\, \hat{\delta}(x - E^t(x'))\psi(x) = \psi[E^{-t}(x)] = \left[(E^{-t})^* \psi\right](x)$$

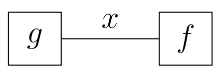
Definition 2.5 (*diagram notation*). vectors, duals, and transforms are conveniently captured by diagrams. We read diagrams from the right to left due to the (unfortunate) order of function application. We represent a vector $f \in \mathcal{H}$ by



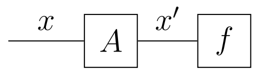
The dual of $g \in \mathcal{H}$ is horizontally flipped



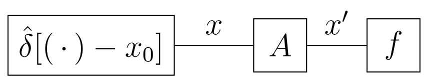
Connecting two lines denotes integration of that variable. The inner product $\langle g|f\rangle$ reads



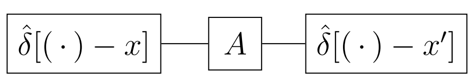
The linear transform $A|f\rangle$ is the vector



The concrete function value $(A\,f)(x_0)$ is equivalent to

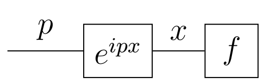


The kernel value $A(x, x')$ is



Example 2.1 (*Fourier transform*). Take $F(x, x') = e^{ixx'}$ and substitute $x \mapsto p, x' \mapsto x$

$$(F\,f)(p) = \int d^n x\, e^{ipx}\, f(x)$$



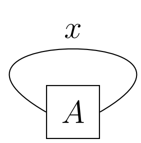
Definition 2.6 (*trace*). The trace of an operator $A(x, x')$ is

$$\mathrm{tr}\,(|x\rangle\langle x'|) = \langle x'|x\rangle$$

Extending by linearity yields

$$\mathrm{tr}A = \int A(x, x)\, dx$$

In diagram notation



In summary, canonical transforms in classical mechanics generate unitaries on the associated Hilbert space. Such unitaries are the pullbacks of the root-densities along a symplectomorphism (which is, in particular, volume-preserving). The converse is not true: most of the unitaries are not of the form 2.1 or, explicitly in terms of kernel, 2.2. The unitaries associated with classical evolution can only permute the underlying domain. The following unitary, for example, cannot be associated with classical evolution

$$(U\,\psi)(x) = \psi(x)e^{i\pi/2}$$

This is different from quantum theory, in which the canonical transforms biject with probability-preserving unitaries. This is one of the major obstacles preventing us from constructing the snapshot closure of classical subsystems using the machinery available to us for discrete, finite quantum systems.

3 Composite systems and mixture

Consider two systems S_1, S_2 with n_1, n_2 degrees of freedom respectively. Denote their phase spaces N_1, N_2, where $\dim N_1 = 2n_1$, $\dim N_2 = 2n_2$. In classical mechanics, the composite system $S = (S_1, S_2)$ has $n_1 + n_2$ degrees of freedom with phase space given by the direct product $N = N_1 \times N_2$. We explore what this construction implies for KvN theory.

Given two distributions $N_1 \xrightarrow{f} \mathbb{R}, N_2 \xrightarrow{g} \mathbb{R}$, the distribution they induce on $N_1 \times N_2 \to \mathbb{R}$ is given by the law of joint probability

$$h(x_1, x_2) = f(x_1)g(x_2)$$

If we represent f, g, h by their square roots $\phi \in \mathcal{H}_1, \varphi \in \mathcal{H}_2, \psi \in \mathcal{H}$, they also satisfy

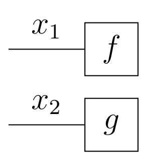
$$\psi(x_1, x_2) = \phi(x_1)\varphi(x_2)$$

The law of joint probability implies that the map $T : \mathcal{H}_1 \times \mathcal{H} \to \mathcal{H}$ sending two subsystem root-distributions to their representation in the composite system is bilinear. The universal representation of this bilinear relation is the tensor product:

Definition 3.1 (*tensor product*). Given two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ over phase spaces N_1, N_2 with $f \in \mathcal{H}_1, g \in \mathcal{H}_2$, their tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ consists of scalar fields on the phase space $N_1 \times N_2$, the tensor product $f \otimes g : N_1 \times N_2 \to \mathbb{C}$ is defined by

$$(f \otimes g)(x_1, x_2) = f(x_1)g(x_2)$$

In our diagram notation, tensor products are read from the top to bottom and simply denoted by vertical juxtaposition



Theorem 3.1 (*universality of the tensor product*). The tensor product $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ is the universal bilinear map. To every Hilbert space $\mathcal{H}_3$ and bilinear map

$$g : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_3$$

There is a unique linear map

$$\tilde{g} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_3$$

such that $\tilde{g}(x_1 \otimes x_2) = g(x_1, x_2)$

The tensor product is our construction to build the kinematics (state description) of composite systems out of subsystems. Its inverse operation is the partial trace.

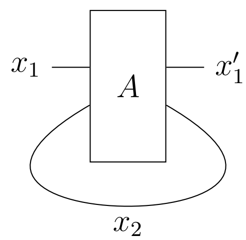
Definition 3.2 (*partial trace*). Given a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the partial trace over $\mathcal{H}_2$ is the linear operator tr_2 defined by

$$\mathrm{tr}_2\left(|x_1\rangle\langle x_1'| \otimes |x_2\rangle\langle x_2'|\right) = \langle x_2|x_2'\rangle |x_1\rangle\langle x_1'|$$

Extending by linearity, the kernel of $\mathrm{tr}_2(A)$ is the following integral whenever well-defined

$$\left[\mathrm{tr}_2(A)\right](x_1, x_1') = \int dx_2\, A(x_1, x_1', x_2, x_2)$$

In diagram notation



Proposition 3.2. The partial trace is trace-preserving: $\mathrm{tr}(A) = \mathrm{tr}_2\left(\mathrm{tr}_1(A)\right)$.

Definition 3.3 (*outer product*). Given $|f\rangle \in \mathcal{H}$, its outer product $\rho_f = |f\rangle\langle f| \in L(\mathcal{H})$ is defined by the kernel

$$\rho_f(x, x') = f(x)^* f(x')$$

Our Hilbert space formalism can be extended by linear combination to account for mixtures. It is important to differentiate between two appearances of probability here: the root-distribution captures stochasticity inherent to the system; for our formalism, this is the initial randomness of a standalone system admitting Hamiltonian description. Mixture probability accounts for marginalized ignorance over an unknown system.

Definition 3.4 (*density operator*). A density operator is a linear operator $\rho \in L(\mathcal{H})$ such that $\mathrm{tr}(\rho) = 1$ and $\forall x, \rho(x, x) > 0$. The space of such operators is denoted $T(\mathcal{H})$.

Proposition 3.3. $\Pr(x) = \rho_f(x, x)$

Proof: Recall our representation $\Pr(x) = |\langle x|f\rangle|^2 = \langle x|f\rangle\langle f|x\rangle = \rho_f(x, x)$

Proposition 3.4. With the interpretation above, $|f\rangle \mapsto U|f\rangle$ is equivalent to $\rho \mapsto U\rho_f U^\dagger$.

Definition 3.5 (*bounded operator*). A linear operator $U \in L(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is bounded if there exists $c \in \mathbb{R}$ such that

$$\forall |f\rangle \in \mathcal{H}, \quad \frac{\|U|f\rangle\|}{\|f\|} \leq c$$

The space of bounded linear operators over $\mathcal{H}$ is denoted $B(\mathcal{H})$.

Proposition 3.5. Every density matrix is bounded and self-adjoint.

Let $A(x_1, x_1', x_2, x_2')$ denote the kernel of a density operator. The resulting kernel of partially tracing A over S_2 is another density operator.

$$[\mathrm{tr}_2(A)](x, x') = \int_{x_2} d^{2n}x A(x_1, x_1', x_2, x_2)$$

Moreover, the spectral theorem tells us that the result of such a partial trace is a continuous ensemble of outer-products of $|\psi_1\rangle \in \mathcal{H}_1$.

Theorem 3.6 (*spectral theorem of bounded operators*[1]). For every bounded self-adjoint operator $A \in B(\mathcal{H})$, there exists a unique projection-valued measure μ^A on the Borel σ-algebra in $\sigma(A)$ with value projections on $\mathcal{H}$, such that

$$\int_{\sigma(A)} \lambda \, d\mu^A(\lambda) = A$$

[1] Brian C. Hall, *Quantum theory for Mathematicians.* Section 7.2, Pg 141

4 From Composite to Subsystems

Consider the Hamiltonian evolution of a standalone composite classical system $S = (S_1, S_2)$

$$E^t_{1,2} : N_1 \times N_2 \to N_1 \times N_2$$

The unitary corresponding to this time-evolution $U^t_{1,2} : \mathcal{H}_{1,2} \to \mathcal{H}_{1,2}$ has kernel

$$U^t_{1,2}(x'_{1,2}, x_{1,2}) = \hat{\delta}(E^t_{1,2}(x_{1,2}) - x'_{1,2})$$

Note that the kernel of the conjugate transpose $U^{t\dagger}_{1,2}$ simply swaps the arguments. Fix $x^0_2 \in N_2$. Consider the composite system with initial density matrix

$$\rho^0_{1,2} = \rho^0_1 \otimes |x^0_2\rangle\langle x^0_2|$$

Tracing over S_2, we have the density operator evolution map $\phi^t_1 : T(\mathcal{H}_1) \to T(\mathcal{H}_1)$

$$\rho^t_1 = \phi^t_1\, \rho^0_1 = \mathrm{tr}_2 \left[U^t_{1,2} \left(\rho^0_1 \otimes |x^0_2\rangle\langle x^0_2| \right) U^{t\dagger}_{1,2} \right]$$

Consider the kernel for this density operator

$$\rho^t_1(x'_1, x_1) = \int dx_2 \int dx''_1\, dx'''_1\, U^t_{1,2} \left(\begin{pmatrix} x'_1 \\ x_2 \end{pmatrix} \leftarrow \begin{pmatrix} x''_1 \\ x^0_2 \end{pmatrix} \right) \rho^0_1(x''_1, x'''_1) U^t_{1,2} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leftarrow \begin{pmatrix} x'''_1 \\ x^0_2 \end{pmatrix} \right) \tag{4.1}$$

We have fixed x^0_2, x_1, x'_1. The integral is effectively over tuples (x_2, x''_1, x'''_1) such that

$$\begin{pmatrix} x'''_1 \\ x^0_2 \end{pmatrix} = E^{-t}_{1,2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x''_1 \\ x^0_2 \end{pmatrix} = E^{-t}_{1,2} \begin{pmatrix} x'_1 \\ x_2 \end{pmatrix}$$

For each x'_1, x_1, find x_2 such that back-evolving both (x_1, x_2) and (x'_1, x_2) projects onto x^0_2. Recall that, having fixed x^0_2, we can define a projection of the Hamiltonian evolution of the composite system onto one of the subsystems $E^t_1 : N_1 \to N_1$

$$E^t_1(x_1) = \pi_2 \left[E^t_{1,2}(x_1, x^0_2) \right]$$

Focus on diagonal elements $x_1 = x'_1$ of equation 4.1 denoting observable probability:

$$\begin{aligned} p^t_1(x_1) &= \int dx_2 \int dx''_1 \rho^0_1(x''_1, x''_1) U^t_{1,2} \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leftarrow \begin{pmatrix} x''_1 \\ x^0_2 \end{pmatrix} \right) \\ &= \int dx_2 \int_{(x''_1, x^0_2) \mapsto (x_1, x_2)} dx''_1\, p^0_1(x''_1) \\ &= \int_{(E^t_1)^{-1}(x_1)} dx''_1 p^0_1(x''_1) \end{aligned}$$

This is reminescent of the pushforward measure of $p^1_0(x)$ along E^t_1, defined by

$$(E^t_{1*}\, p^0_1)(U) = p^0_1 \left[(E^t_1)^{-1}(U) \right]$$

Since the pushforward measure E_{1*}^t defines a map from probability distributions to probability distributions, we may extend it to a map on $\mathcal{H}_1$ defined by

$$\left(E_{1*}^t\,\psi\right)(x) = \left(\int_{(E_1^t)^{-1}(x)} dx'\,|\psi(x')|^2\right)^{1/2}$$

The pushforward $E_{1*}^t : \mathcal{H}_1 \to \mathcal{H}_1$ of root-probability is not a surjection (only maps to real-valued root-densities) so is apparantly not a unitary nor a useful definition. We can see that only volume-preserving endomorphisms of phase space define useful unitaries on $\mathcal{H}$.

Remark 4.1. The phase space evolution of a subsystem only tell us how the diagonal entries of the density matrix evolves. This under-determines the density matrix evolution we have obtained in equation 4.1. In classical theory, the time-evolution of a subsystem is given by a one-parameter family of phase space endomorphisms. In quantum theory, the time-evolution of an open system is given by a one-parameter family of density operator endomorphisms; the off-diagonal entries needs to be explicitly specified since they are physically observable. We now explore whether the off-diagonal evolution of the density matrix can be solely obtained from the diagonal. If not, then only knowing the dynamics E_1^t of a subsystem yields less information than the density operator map ϕ_1^t we obtained by tracing the dynamics of the composite system.

5 Coupled Harmonic Oscillator

Phase space evolution of a coupled harmonic oscillator system is given by

$$x_{1,2}^t = \begin{pmatrix} q_1(t) \\ p_1(t) \\ q_2(t) \\ p_2(t) \end{pmatrix} = \begin{pmatrix} \alpha(t) & \beta(t) & \gamma(t) & \delta(t) \\ \epsilon(t) & \alpha(t) & \phi(t) & \gamma(t) \\ \gamma(t) & \delta(t) & \alpha(t) & \beta(t) \\ \phi(t) & \gamma(t) & \epsilon(t) & \alpha(t) \end{pmatrix} \begin{pmatrix} q_1(0) \\ p_1(0) \\ q_2(0) \\ p_2(0) \end{pmatrix} = \begin{pmatrix} \Theta & \Theta' \\ \Theta' & \Theta \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = E_{1,2}^t(x_{1,2}^0)$$

The linear relation and block structure is particular for this system. The coefficients are

$$\begin{aligned}
\alpha(t) &= \frac{1}{2}\left(\cos\omega_f t + \cos\omega_s t\right) \\
\beta(t) &= \frac{1}{2m}\left(\frac{\sin\omega_f t}{\omega_f} + \frac{\sin\omega_s t}{\omega_s}\right) \\
\gamma(t) &= \frac{1}{2}\left(\cos\omega_s t - \cos\omega_f t\right) \\
\delta(t) &= \frac{1}{2m}\left(\frac{\sin\omega_s t}{\omega_s} - \frac{\sin\omega_f t}{\omega_f}\right) \\
\epsilon(t) &= -\frac{m}{2}\left(\omega_f \sin\omega_f t + \omega_s \sin\omega_s t\right) \\
\phi(t) &= \frac{m}{2}\left(\omega_f \sin\omega_f t - \omega_s \sin\omega_s t\right)
\end{aligned}$$

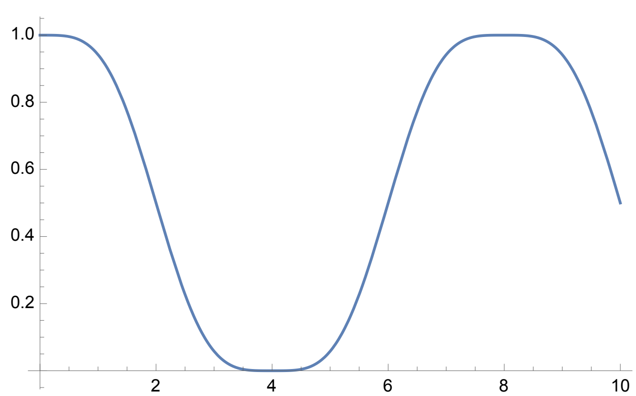
Here $\omega_s = \sqrt{\frac{k}{m}}, \omega_f = \sqrt{\frac{k+2\kappa}{m}}$ are the normal modes. Let $x_2^0 = (0,0)$, the projected time-evolution on the first system is

$$x_1^t = \pi_1\left[E_{1,2}^t(x_1^0, x_2^0)\right] = \Theta x_1^0$$

As expected, phase space volume of the first system is not conserved

$$\det E_1^t = \frac{1}{4}\left(\frac{\left(\omega_f^2 + \omega_s^2\right)\sin\left(\omega_f t\right)\sin\left(\omega_s t\right)}{\omega_f \omega_s} + 2\cos\left(\omega_f t\right)\cos\left(\omega_s t\right) + 2\right)$$

Take $\omega_s = \pi/4, \omega_f = \pi/2$, the determinant of $E_1^t = \Theta$ looks like



We consider the density matrix operator in equation 4.1

$$\rho_1^t(x_1', x_1) = \int dx_2 \int dx_1'' \, dx_1''' \, U_{1,2}^t \left(\begin{pmatrix} x_1' \\ x_2 \end{pmatrix} \leftarrow \begin{pmatrix} x_1'' \\ x_2^0 \end{pmatrix} \right) \rho_1^0(x_1'', x_1''') U_{1,2}^t \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leftarrow \begin{pmatrix} x_1''' \\ x_2^0 \end{pmatrix} \right)$$

To evaluate the integral for $x_2^0 = 0$, substitute the expression for $U_{1,2}^t$

$$\rho_1^t(x_1', x_1) = \int dx_2 \int dx_1'' \, dx_1''' \, \hat{\delta} \begin{pmatrix} \Theta x_1'' - x_1' \\ \Theta' x_1'' - x_2 \end{pmatrix} \hat{\delta} \begin{pmatrix} \Theta x_1''' - x_1 \\ \Theta' x_1''' - x_2 \end{pmatrix} \rho_1^0(x_1'', x_1''')$$

A special quantity of interest here is

$$\det \Theta' = \det \begin{pmatrix} \gamma & \delta \\ \phi & \gamma \end{pmatrix} = \frac{1}{4} \left(-\frac{\left(\omega_f^2 + \omega_s^2\right) \sin\left(t\omega_f\right) \sin\left(t\omega_s\right)}{\omega_f \omega_s} - 2\cos\left(t\omega_f\right)\cos\left(t\omega_s\right) + 2 \right)$$

Note that $\det \Theta' + \det \Theta = 1$. The integral for density operator evolution has different behavior based on the singularity of Θ', depending on t.

- When $\det \Theta' \neq 0$, we have $x_2 = \Theta' x_1'' = \Theta' x_1''' \implies x_1'' = x_1'''$. Then $x_1 = \Theta x_1''' = \Theta x_1'' = x_1'$, so only diagonal terms are nonzero.
- When $\det \Theta' = 0 \iff \det \Theta = 1$, we have $x_1'' = \Theta^{-1} x_1', x_1''' = \Theta^{-1} x_1$. We also need

$$x_2 = \Theta' \Theta^{-1} x_1' = \Theta' \Theta^{-1} x_1$$

Let δ denote the indicator function which is 1 at the origin and zero elsewhere. The closed form of the density matrix operator evolution reads

$$\rho_1^t(x_1', x_1) = \begin{cases} \rho_1^0(\Theta^{-1} x_1', \Theta^{-1} x_1), & x_1 - x_1' \in \ker \Theta' \Theta^{-1} \\ \delta(x_1' - x_1)(E_{1*}^t \, p_1^0)(x_1), & \text{otherwise} \end{cases}$$

Recall that given an initial configuration $x_1^0 \in N_1$, the quantity $\Theta x_1^0 \in N_1$ is the configuration of the subsystem after $(x_1^0, x_2^0 = 0)$ evolves for time t. Similarly, $\Theta' x_1^0 \in N_2$ is the configuration of the ancilla after $(x_1^0, 0)$ evolves for time t. When the phase space volume of the subsystem is conserved, the density operator evolution is nonzero for pairs $x_1^t, (x_1^t)'$ for which initial conditions defined by $x_1^0 = \Theta^{-1} x_1^t, x_1^{0'} = \Theta^{-1}(x_1^t)'$ evolve to the same configuration for the ancilla. While ρ_1^t contains no information about the ancilla's configuration, by examining the appearance of off-diagonal entries we can infer which points x_1, x_1' map to the same configuration of the ancilla under time-evolution. The traced-out knowledge of the whole obtains more information than the observable knowledge of the part. It is hypothesized, but yet to be confirmed, that this is general behavior of the integral 4.1.

6 Appendix

6.1 Hamiltonian mechanics and Liouville's theorem

This section reviews some concepts from Hamiltonian mechanics and Liouville's theorem, which provides the link from Hamiltonian mechanics to Koopman von Neumann theory.

Consider a system S with finite degrees of freedom. Denote its configuration manifold M and phase space $N = T^*M$. We begin with the canonical symplectic form on N.

$$\omega = dq \wedge dp = \sum dq_i \wedge dp_i$$

Definition 6.1 (*Hamiltonian vector field*). A smooth scalar function $f : N \to \mathbb{R}$ induces a Hamiltonian vector field $X_f : N \to TN$ defined by the equation, for all vector field Y:

$$df\, Y = \omega(X_f \otimes Y)$$

Equivalently, in terms of contractions

$$df = \iota_{X_f}\, \omega$$

Let $Y = \partial_{q_i}$ in local coordinates we have

$$df\, \partial_{q_i} = \partial_{q_i} f = \left(-\sum dp_i \wedge dq_i\right) (X_f \otimes dq_i) = -X_f\, dp_i$$

Similarly, let $Y = \partial_{p_i}$ yields

$$df\, \partial_{p_i} = \partial_{p_i} f = \left(\sum dq_i \wedge dp_i\right) (X_f \otimes dp_i) = X_f\, dq_i$$

This gives local coordinates expression for X_f in terms of f:

$$X_f = (\partial_{p_i} f)\ \partial_{q_i} - (\partial_{q_i} f)\ \partial_{p_i}$$

Definition 6.2 (*Poisson bracket*). the Poisson bracket of smooth fields $f, g : N \to \mathbb{R}$ is

$$\{f, g\} = X_g\, f = (\partial_{q_i} f)\, (\partial_{p_i} g) - (\partial_{p_i} f)\, (\partial_{q_i} g)$$

Given a scalar function f, its Hamiltonian vector field X_f generates a one-parameter family of automorphisms $e^{tX_f} : N \to N$ which flows a point p along X_f for time t. The exponential notation is so defined since, for $p(t) = e^{tX_f} p_0$,

$$\partial_t p(t) = X_f\, p(t)$$

Definition 6.3 (*canonical transformation*). A canonical transformation $N \xrightarrow{\varphi} N$ is an automorphism of phase space which preserves ω under pullback

$$\varphi^* \omega = \omega$$

In local coordinates with J_φ denoting the Jacobian and Γ the matrix representation of ω in basis $\{\nabla_q, \nabla_p\}$, this is equivalent to

$$J_\varphi^T \Gamma J_\varphi = \Gamma$$

Theorem 6.1 (*Hamiltonian evolutions are canonical transforms*). Let X_f be a Hamiltonian vector field generated by f. For every t for which time-evolution e^{tX_f} is defined

$$\omega = e^{-tX_f *}\omega$$

Proof. We first show that $\mathcal{L}_{X_f}\,\omega = 0$. By Cartan's formula

$$\mathcal{L}_{X_f}\,\omega = d\left(\iota_{X_f}\omega\right) + \iota_{X_f}\,d\omega$$

The second term vanishes by ω symplectic (thus closed), while the first term

$$d\left(\iota_{X_f}\omega\right) = d(df) = 0$$

Consider the function $F(t) = e^{-tX_f *}\omega$. The Lie derivative is defined as

$$\mathcal{L}_{X_f}\omega = \partial_t F(t)$$

Then $\partial_t F(t) = 0$ for all t implies $F(t) = F(0) = \omega$.

Definition 6.4 (*volume form on phase space*). Let $T^*M = N$ have dimension $2n$. The nth exterior product $\Omega = \omega^{\wedge n}$ defines a nonvanishing volume form on N. In local coordinates

$$\bigwedge_{i=1}^{n}\omega = \bigwedge_{i=1}^{n}\left(\sum_{j=1}^{n} q_j \wedge p_j\right) = \bigwedge_{i=1}^{n}(q_i \wedge p_i)$$

Theorem 6.2 (*Liouville's theorem*). For every scalar Hamiltonian H, time-evolution e^{tX_H} conserves phase space volume.

$$e^{tX_H *}\Omega = \Omega$$

Proof. exterior product commutes with pullback: $\varphi^*(a \wedge b) = \varphi^* a \wedge \varphi^* b$, and $\Omega = \omega^{\wedge n}$.

In the remaining parts of this section, we explore discrete systems with finite degrees of freedom. We show that every such system evolving under a completely positive trace preserving map admits a snapshot dilation to a composite system with dimension no more than n^3. This result relies on a clever application of Stinespring dilation theorem. We also show that the probability evolution of every classical system may be regarded as the effect of a cptp map.

6.2 Stinespring dilation theorem

The theorem below is adopted from [**?**], theorem 4.1.

Theorem 6.3 (*Stinespring dilation theorem*). Given a unital C^* algebra $\mathcal{A}$. Let $\phi : \mathcal{A} \to B(\mathcal{H})$ be a completely positive map. There exists a Hilbert space $\mathcal{K}$, unital $*$-homomorphism $\pi : \mathcal{A} \to B(\mathcal{K})$, and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ with $\|\phi(1)\|^2 = \|V\|^2$ such that

$$\phi(a) = V^*\pi(a)V$$

For our purposes take $\mathcal{A} = B(\mathcal{H})$. Consider the algebraic tensor product $B(\mathcal{H}) \otimes \mathcal{H}$ with inner product extended linearly from

$$\langle \rho \otimes \psi_1, \varrho \otimes \psi_2 \rangle \equiv \langle \phi(\varrho^\dagger \rho)\psi_1, \psi_2 \rangle$$

That ϕ is completely positive ensures that $\langle \ , \ \rangle$ is positive semidefinite since

$$\langle \sum \rho_j \otimes \psi_j, \sum \rho_i \otimes \psi_i \rangle = \sum_{i,j} \langle \phi(\rho_i^\dagger \rho_j)\psi_i, \psi_j \rangle = \langle \phi^n \left[(\rho_i^\dagger \rho_j) \right] (\psi_i), (\psi_j) \rangle$$

Here $\phi^n \left[(\rho_i^\dagger \rho_j) \right]$ is a matrix with ϕ applied element-wise. In particular, the positive semidefinite bilinear form satisfies Cauchy-Schwarz inequality,

$$|\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle$$

The kernel $\mathcal{N}$ of the quadratic map $u \mapsto \langle u, u \rangle$ is a subspace of $B(\mathcal{H}) \otimes \mathcal{H}$

$$\ker (u \mapsto \langle u, u \rangle) = \{ u \in B(\mathcal{H}) \otimes \mathcal{H} \mid \forall v \in B(\mathcal{H}) \otimes \mathcal{H}, \langle u, v \rangle = 0 \}$$

The induced bilinear form on $B(\mathcal{H}) \otimes \mathcal{H}/\mathcal{N}$ is an inner product

$$\langle u + \mathcal{N}, v + \mathcal{N} \rangle = \langle u, v \rangle$$

Consider the map $\pi : B(\mathcal{H}) \to \mathrm{End}\,(B(\mathcal{H}) \otimes \mathcal{H})$ by

$$\pi_\rho \left(\sum \varrho_i \otimes \psi_i \right) = \sum (\rho \varrho_i) \otimes \psi_i$$

The following inequality in $M_n(B(\mathcal{H}))^+$ is satisfied

$$(\varrho_i^\dagger \rho^\dagger \rho \varrho_j) \leq \|\rho^\dagger \rho\| \cdot (\varrho_i^\dagger \varrho_j)$$

Consequently,

$$\begin{aligned}
\langle \pi_\rho \sum \varrho_j \otimes \psi_j, \pi_\rho \sum \varrho_i \otimes \psi_i \rangle &= \sum_{ij} \langle \phi(\varrho_j^\dagger \rho^\dagger \rho \varrho_j)\psi_j, \psi_i \rangle \\
&\leq \|\rho^\dagger \rho\| \sum_{i,j} \langle \phi(\varrho_i^\dagger \varrho_j)\psi_j, \psi_i \rangle \\
&= \|\rho\|^2 \langle \sum \varrho_j \otimes \psi_j, \sum \varrho_i \otimes \psi_i \rangle
\end{aligned}$$

Then π_ρ renders $\mathcal{N}$ invariant and induces a quotient linear transformation $\pi : B(\mathcal{H}) \to \mathrm{End}(B(\mathcal{H}) \otimes \mathcal{H}/\mathcal{N})$. Let $\mathcal{K}$ denote the completion of $B(\mathcal{H}) \otimes \mathcal{H}/\mathcal{N}$. The inequality above also shows that $\|\pi_\rho\| \leq \|\rho\|$ so it extends to a linear operator on $\mathcal{K}$. One can also show that $\pi : B(\mathcal{H}) \to B(\mathcal{K})$ is a unital $*$-homomorphism. Define $V : \mathcal{H} \to \mathcal{K}$ by

$$V(\psi) = 1_{B(\mathcal{H})} \otimes \psi + \mathcal{N}$$

Now $\|V\|^2 = \|\phi(1)\|$ since

$$\|V(\psi)\|^2 = \langle 1 \otimes \psi, 1 \otimes \psi \rangle = \|\phi(1)\| \cdot \|\psi\|^2$$

To complete the proof, observe that

$$\begin{aligned}
\langle V^\dagger \pi_\rho V \psi_1, \psi_2 \rangle &= \langle \pi_\rho(1 \otimes \psi_1 + \mathcal{N}), 1 \otimes \psi_2 + \mathcal{N} \rangle \\
&= \langle \rho \otimes \psi_1, 1 \otimes \psi_2 \rangle \\
&= \langle \phi(\rho)\psi_1, \psi_2 \rangle
\end{aligned}$$

When ϕ is unital, V is an isometry, then $\mathcal{H}$ may be identified as a subspace of $\mathcal{K}$.

6.3 Finite snapshot closure of quantum systems

Reference for this section is *Physical realizations of quantum operations.* We begin by considering the process which gives rise to quantum subsystems: beginning with the unitary evolution of a composite system and tracing out a subsystem. Consider a system-environment evolving according to U^t_{ab}.

$$\rho^t_{ab} = U_{ab}\rho^0_{ab}U^\dagger_{ab}$$

Assuming that $\rho^0_{ab} = \rho^0_a \otimes |0\rangle\langle 0|$ and tracing out b, we have

$$\rho^t_a = \mathrm{tr}_b \left[U_{ab}\left(\rho^0_a \otimes |0\rangle\langle 0|\right)U^\dagger_{ab}\right] = \sum_b E_k\rho^0_a E^\dagger_k$$

Here the $E_k = \langle k|U_{ab}|0\rangle$ are the Kraus operators defined by $E_k = \langle k|U_{ab}|0\rangle$. By unitarity

$$\sum_k E^\dagger_k E_k = \sum \langle 0|U^\dagger_{ab}|k\rangle\langle k|U_{ab}|0\rangle = I_a$$

Let $T(\mathcal{H})$ denote the trace-class operators over our Hilbert space $\mathcal{H}$. This defines a completely positive trace preserving map $\phi : T(\mathcal{H}) \to T(\mathcal{H})$:

$$\phi(\rho) = \sum_k E_k\rho E^\dagger_k \tag{6.1}$$

This is the perspective from whole to part. We are interested in the converse, that such Kraus form is equivalent to a system being a subsystem. In other words, a completely positive trace-preserving map of the form 6.1 such that

$$\sum_k E^\dagger_k E_k = I \tag{6.2}$$

may be embedded the subsystem evolution of a larger quantum system.

We first note that every such quantum operation may be equivalently defined in terms of its dual $\bar{\phi} : B(\mathcal{H}) \to B(\mathcal{H})$ acting on observables

$$\mathrm{tr}\left[O\,\phi(\rho)\right] = \mathrm{tr}\left[\bar{\phi}(O)\,\rho\right] \tag{6.3}$$

This equality gives us the explicit form of $\bar{\phi}$ as

$$\bar{\phi}(O) = \sum_k E^\dagger_k O E_k$$

The trace-preserving property of ϕ translates to $\bar{\phi}$ being unital, since

$$\bar{\phi}(I) = \sum_k E^\dagger_k I E_k = I$$

Invoking the Stinespring dilation theorem we have

$$\bar{\phi}(O) = (1_{B(\mathcal{H})} \otimes -)^\dagger(O \otimes 1_{\mathcal{H}})(1_{B(\mathcal{H})} \otimes -) \tag{6.4}$$

Here $\bar{\phi}$ is dilated to a linear operator $(- \otimes 1_{\mathcal{H}})$ on the Hilbert space $B(\mathcal{H}) \otimes \mathcal{H}$. Its projection onto the subspace which annihilates the kernel of the following bilinear form on $B(\mathcal{H}) \otimes \mathcal{H}$ is a unitary operator

$$\langle |i\rangle\langle j| \otimes \psi, |i'\rangle\langle j'| \otimes \psi' \rangle = \sum_k \langle (E_k^\dagger |j'\rangle\langle i'|i\rangle\langle j| E_k)\psi, \psi' \rangle = \delta_{i,i'} \langle \bar{\phi}(|j'\rangle\langle j|)\psi, \psi' \rangle$$

On this Hilbert space (up to quotient), we have

$$\bar{\phi}(O) = E^\dagger [O \otimes I_{\mathcal{H}}] E$$

We can use the inner product relation, together with Gram-Schmidt, to define the standard representation of the Hilbert space given by the dilation. Alternatively, note that

$$B(\mathcal{H}) \xrightarrow{E} B(\mathcal{H}) \otimes \mathcal{H} = \sum_k E_k \otimes |k\rangle$$

Matching the operator-sum form of $\bar{\phi}$ in the standard Hilbert space with equation 6.4 in the dilated space with the nonstandard inner product gives us the standard representation of the Stinespring isometry E

$$\bar{\phi}(O) = E^\dagger [O \otimes I_{\mathcal{H}}] E = \sum_k E_k^\dagger O E_k$$

Recall equation 6.3 to obtain the expression for $\phi(\rho) = \mathrm{tr}_k(E\rho E^\dagger)$

$$\begin{aligned} \mathrm{tr}\left[E^\dagger (O \otimes I_{\mathcal{H}}) E\, \rho\right] &= \sum_k \mathrm{tr}\left[O\ \mathrm{tr}_k\left((E_k \otimes |k\rangle)\rho(E_k^\dagger \otimes \langle k|)\right)\right] \\ &= \mathrm{tr}\left[O\, \mathrm{tr}_k(E\rho E^\dagger)\right] = \mathrm{tr}(O\, \phi(\rho)) \end{aligned}$$

Since $E : B(\mathcal{H}) \otimes \mathcal{H} \to B(\mathcal{H})$ is an isometry, we may dilate it to a unitary such that

$$E\rho E^\dagger = U(\rho \otimes |0\rangle\langle 0|)U^\dagger$$

Then U constitutes a snapshot dilation of ϕ:

$$\phi(\rho) = \sum E_k \rho E_k^\dagger = \mathrm{tr}_k(E\rho E^\dagger) = \mathrm{tr}_k\left[U(\rho \otimes |0\rangle\langle 0|)U^\dagger\right]$$

6.4 Closure of stochastic evolution

Given a n-dimensional stochastic matrix γ. It defines a completely-positive trace-preserving map $\phi : T(\mathcal{H}) \to T(\mathcal{H})$ by

$$\phi(\rho) = \sum E_{ij} \rho E_{ij}^\dagger, \quad E_{ij} = \sqrt{\gamma_{ji}} |i\rangle\langle j|$$

Note the order of i, j in the definition. The Kraus operators $\{E_{ij}\}$ are positive and satisfy the completeness relation 6.2.

$$\sum_{ij} E_{ij}^\dagger E_{ij} = \sum_{ij} \gamma_{ji} |j\rangle\langle i|i\rangle\langle j| = \sum_j \left(\sum_i \gamma_{ji}\right) |j\rangle\langle j| = \sum_j |j\rangle\langle j| = 1$$

This Kraus decomposition is not, in general, minimal. The trace-preserving map ϕ is also not the unique quantum channel whose observable effect is equivalent to γ. Every system with n configurations may be regarded as part of a unitary evolution of a quantum system with no more than n^3 configurations.